

# Viscoelastic-electromagnetism and Hall viscosity

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## Abstract

We introduce a kind of electromagnetism, which we call viscoelastic-electromagnetism, to investigate viscoelastic transport phenomena. It is shown that Cartan's formalism of general relativity is essential for viscoelastic theory, and then the corresponding electric and magnetic fields are regarded as a velocity gradient and a Burgers vector density, respectively. As an application of this formalism, the Středa formula for the Hall viscosity is obtained.

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## 1 Introduction

According to the linear response theory, we can obtain a number of quantities characterizing transport phenomena by calculating the correlation functions in perturbation regime. In particular, charge transport is the most tractable not only for theoretical calculation, but also experimental manipulation. Indeed, the charge current is simply given by introducing the interaction between matter fields and U(1) electromagnetic gauge field, and then differentiating the action with respect to the gauge potential,  $J^\mu = \delta S / \delta A_\mu$ . In this sense, it seems difficult to formulate thermal transport in a similar manner, because we have to find the corresponding *gauge potential* to be coupled with matter fields.

While we have to involve the charge conservation, namely the particle number conservation law to investigate charge transport, we have to deal with the energy conservation in the case of thermal transport phenomena. Therefore, it is natural to utilize the framework to describe the energy itself, namely general relativity. Luttinger, actually, showed that the gravitational potential is essential for thermal transport [1]. After that it turns out that we can deal with thermal transport in a quite similar manner by introducing a kind of electromagnetism, called gravito-electromagnetism [2]. In this formalism the gauge potential is coming from a part of degrees of freedom in the spacetime geometry, and then the thermal gradient can be represented as the corresponding electric field. Based on this formalism, various interesting phenomena can be described by studying the gravitational counterpart of electromagnetism, for example, the surface state of the topological insulator/superconductor [3, 4, 5], the Středa formula for the thermal Hall conductivity [6], the thermal analog of the chiral magnetic effect [7], and so on.

In this paper, we explore a possibility of such an effective electromagnetic description of viscoelastic theory. In other words, we have to find the corresponding gauge potential-like degrees of freedom to this case. In viscoelastic transport phenomena we essentially investigate transport of momentum. Thus, we have to deal with the theory of momentum: it is again general relativity. We would like to show that the vielbein, introduced in Cartan's formalism

for general relativity, can be interpreted as such a gauge potential describing viscoelastic theory, and then investigate some aspects of the corresponding electromagnetism, which we will call *viscoelastic-electromagnetism*. In this formalism the 2-form curvature, the field strength tensor of the corresponding electromagnetism, is just given by torsion tensor. Its electric and magnetic components represent the gradient of velocity and the Burgers vector density, respectively.

Based on this formalism, we then investigate dissipationless transport in viscoelastic theory. Such a transport phenomenon is characterized by the Hall viscosity [8, 9, 10, 11, 12, 13, 14, 15, 16].<sup>5</sup> This quantity is only observed when the time reversal symmetry of the system is broken, as well as the Hall conductivity. We will show that, simply generalizing the result for charge transport, we can obtain the Středa formula for the Hall viscosity [18, 19]: the Hall coefficient can be represented in terms of thermodynamic quantities.

This paper is organized as follows. In Sec. 2, we investigate the generic formalism of electromagnetism with emphasis on its relation to general relativity. Starting with gravito-electromagnetism as a review, we formulate viscoelastic-electromagnetism from the vielbein as a fundamental degree of freedom. We will then clarify the meanings of the electric and magnetic fields, the gauge transformation, etc, in terms of viscoelastic theory. In Sec. 3 we derive the Středa formula for the Hall viscosity by applying viscoelastic-electromagnetic formalism. Introducing the magnetization corresponding to viscoelastic-magnetic field, we show the Hall viscosity can be expressed in terms of thermodynamic quantities. Section 4 is devoted to a summary and discussion.

## 2 Gravitational electromagnetism

In this section, we investigate two types of electromagnetisms, which are coming from general relativity. One of them is gravito-electromagnetism, used for thermal transport, and the other is viscoelastic-electromagnetism, for viscoelastic theory. After reviewing the former one, we will give definitions and detailed discussions for the latter.

### 2.1 Gravito-electromagnetism

We would like to show that how gravito-electromagnetism is derived from general relativity [2]. In particular, when the temporal direction is compactified on  $S^1$  with circumference  $\beta = 1/T$ , we can describe thermal transport by using such an induced U(1) gauge field [3, 6, 20]. Thus, this gravito-electromagnetism is regarded as the U(1) gauge theory coming from the Kaluza-Klein mechanism.

The 4-dimensional metric is generally written in the following form,

$$ds^2 = \gamma_{ij}dx^i dx^j + g_{00}(dx^0 + A_i dx^i)^2, \quad (2.1)$$

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<sup>5</sup>A similar approach to the Hall viscosity has been quite recently proposed in [17].

where we parametrize

$$A_i = \frac{1}{g_{00}}g_{0i}, \quad \gamma_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}. \quad (2.2)$$

We will see that gravito-electromagnetism is formulated based on this 3-dimensional 1-form as the U(1) gauge field. Then we introduce the following parametrization,

$$g_{00} = -e^{2\sigma}. \quad (2.3)$$

This  $\sigma$  is called *dilaton*, which plays a role as the scalar potential for gravito-electromagnetism. In fact, when the temporal direction is compactified on  $S^1$  with circumference  $\beta = 1/T$ , this part of the metric is proportional to  $\beta^2$ , so that the temperature dependence of the dilaton yields  $\sigma = -\log(T/T_0)$ , where  $T_0$  is the temperature at thermal equilibrium. Thus, the thermal gradient is given by

$$\nabla_i \sigma = -\frac{\nabla_i T}{T}. \quad (2.4)$$

Since the right hand side of this equation is just regarded as the electric field of gravito-electromagnetism,  $E_i = -\nabla_i T/T$ , the dilaton is nothing but the scalar potential for gravito-electromagnetism. Note that this gravito-electric field is also represented as the temporal derivative of the vector potential in Eq. (2.2), and the gravito-magnetic field is similarly introduced as

$$B^i = \epsilon^{ijk} \partial_j A_k, \quad (2.5)$$

where  $\epsilon^{ijk} \equiv \epsilon^{0ijk} e^{-\sigma}$  with  $\epsilon^{0123} = 1/\sqrt{-g}$ . This quantity is essentially related to the rotation of the system [2]. Indeed the conjugate quantity to this magnetic field, namely the corresponding magnetization, yields the angular-momentum operator [6],

$$M_i = \epsilon_{ijk} x^j T^{k0} = L_i, \quad (2.6)$$

where  $T^{\mu\nu}$  stands for the stress tensor.

The gauge transformation for this U(1) connection is coming from the higher-dimensional coordinate group  $x'^0 = x^0 + \lambda(x^i)$ ,

$$A'_i = A_i - \partial_i \lambda. \quad (2.7)$$

Thus, we can define the conserved current corresponding to this U(1) symmetry. Since this is coming from the conservation law of energy, we can deal with thermal transport with this formalism.

## 2.2 Viscoelastic-electromagnetism

In this section, we formulate another kind of electromagnetism, *viscoelastic-electromagnetism*, in order to deal with viscoelastic theory from the field theoretical point of view. Viscoelastic transport is based on momentum conservation, while electric and thermal ones are coming from carrier and energy conservations, respectively. Therefore, we have to introduce degrees

of freedom describing momentum transport with manifesting its conservation. We would like to show that *vielbein* plays a fundamental role in constructing viscoelastic-electromagnetism.

Let us then start with the vielbein formalism of the curved spacetime, which is also called *Cartan's formalism*. The metric of the spacetime is generally written by using vielbein  $e_\mu^a$ , which connects the global curved spacetime coordinate with the locally flat coordinate,

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (2.8)$$

where  $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$  is the locally flat Minkowski metric. In this formalism each coordinate is dealt with separately. Thus, this vielbein can be regarded as a 1-form with respect to the global curved spacetime coordinate.

We can provide a physical meaning for the vielbein in viscoelastic theory. The time component of the vielbein  $e_\mu^0$  coincides with the gauge fields of gravito-electromagnetic fields  $e_\mu^0 = (e^\sigma, e^\sigma A_i)$  discussed in the previous section. For the spatial component, introducing the displacement vector  $u^i$ , which is regarded as 0-form, the vielbein can be expanded as  $e_\mu^i = \delta_\mu^i + \partial_\mu u^i$  in the linear order [21]. Here  $\partial_\mu u^i \equiv u_\mu^i$  is the distortion tensor, which is just 1-form with respect to the global coordinate. Thus, the vielbein plays almost the same role as this distortion tensor. Therefore, we define electromagnetism corresponding to this 1-form as *viscoelastic-electromagnetism*. The 2-form, obtained from the vielbein, yields the torsion tensor,

$$T^a = de^a + \omega_b^a \wedge e^b. \quad (2.9)$$

In the following we assume the spin connection is zero for simplicity. Thus, the field strength 2-form is given by

$$F_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a. \quad (2.10)$$

If this 2-form takes non-zero value, the displacement vector, namely the vector potential for viscoelastic-electromagnetism, cannot be single-valued. Actually, the corresponding magnetic field is given by

$$B^{ia} = \epsilon^{ijk} \partial_j e_k^a, \quad (2.11)$$

and the magnetic flux

$$b^a = \int d\vec{S} \cdot \vec{B}^a, \quad (2.12)$$

is identified as the *Burgers vector*, which characterizes a lattice dislocation in a crystal. Similarly, the electric field yields Eq. (2.4) and

$$E_i^j \equiv F_{0i}^j = -\partial_i e_0^j \equiv -\partial_i v^j, \quad (2.13)$$

where we defined the spatial derivative of the vielbein as the gradient of the velocity field due to the relation between the vielbein and the distortion fields. This shows the velocity can be interpreted as the chemical potential for viscoelastic-electromagnetism, and also that we can formulate the momentum transport response by applying this viscoelastic-electric field.

We can show that the gauge transformation is consistent with viscoelastic theory. Taking the transformation for the 1-form,  $e_\mu^a \rightarrow e_\mu^a + \partial_\mu w^a$ , it does not affect the torsion tensor 2-form because we have  $\epsilon^{\mu\nu} \partial_\mu \partial_\nu w^a = 0$ . This symmetry is coming from the conservation of energy and momentum, i.e., translation symmetry, and thus, we can derive the corresponding momentum current as Nöther's current.

Let us comment on the topological action, which plays an important role in dissipationless transport. The  $(2+1)$ -dimensional topological action for viscoelastic theory is constructed in a similar way to the Chern-Simons action [12],

$$S = \frac{1}{32\pi} \frac{2}{\ell^2} \int d^3x (\det e) \eta_{ab} \epsilon^{\mu\nu\rho} e_\mu^a \partial_\nu e_\rho^b. \quad (2.14)$$

The covariant form of this action is given by  $S \sim \int e^a \wedge T_a$ . Note that there exists a dimensionful constant  $\ell$  in this topological action. This factor makes the topological origin of this action ambiguous.<sup>6</sup> The energy-momentum tensor reads

$$T_a^\mu = \frac{\hbar}{8\pi\ell^2} \epsilon^{\mu\nu\rho} \partial_\nu e_{\rho a}. \quad (2.15)$$

The viscoelastic transport coefficient, the Hall viscosity, is simply obtained from the corresponding Chern-Simons action (2.14),

$$\eta_H = \frac{\hbar}{8\pi\ell^2}. \quad (2.16)$$

Identifying the dimensionful constant as the magnetic length  $1/\ell_B = \sqrt{eB/\hbar}$ , this reproduces well-known result. The contribution of the energy current to the thermal Hall conductivity can be obtained from Eq. (2.15). In general, the thermal-conductivity tensor is defined as  $Q^i = T^i_0 - \mu J^i = -\kappa^{ij} \partial_j T$ , where  $Q^i$ ,  $\mu$ , and  $J^i$  are the heat current, the chemical potential, and the charge current, respectively. Therefore, the contribution from  $T^i_0$  to the thermal Hall conductivity is  $\hbar T/(8\pi\ell^2)$ , which is proportional to the Hall viscosity.

Similarly, the  $(3+1)$ -dimensional topological action, the viscoelastic analog of the  $\theta$ -term, is given by

$$S = \frac{1}{32\pi^2} \frac{2}{\ell^2} \int d^4x (\det e) \theta(x) \eta_{ab} \epsilon^{\mu\nu\rho\sigma} \partial_\mu e_\nu^a \partial_\rho e_\sigma^b. \quad (2.17)$$

The covariant form of this action is directly related to Nieh-Yan term,  $T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b$  [26]. In terms of viscoelastic-electromagnetism the Lagrangian is written as an inner product of electric and magnetic fields,

$$S = \frac{1}{32\pi^2} \frac{2}{\ell^2} \int d^4x (\det e) \theta(x) \eta_{ab} \vec{E}^a \cdot \vec{B}^b. \quad (2.18)$$

When the  $\theta$ -angle has spatial dependence, we observe the Hall viscosity on the domain-wall. In particular, in the time reversal symmetric system, the  $\theta$ -angle has to be  $\theta = 0$  or  $\pi$ . Therefore, this  $\theta$ -term is reduced to the Chern-Simons action (2.14) on the boundary of the

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<sup>6</sup>See, for example, [22, 23, 24, 25].

topological insulator [27, 28] because the gradient of the  $\theta$ -angle is proportional to  $\delta$ -function. On the other hand, applying the temporal dependence of the  $\theta$ -angle, namely  $\dot{\theta} \neq 0$ , the momentum current is proportional to viscoelastic-magnetic field as  $J_a^i (= T_a^i) \sim (\dot{\theta}/\ell^2) B_a^i$ . This is just the viscoelastic analog of the chiral magnetic effect [7].

### 3 Středa formula for Hall viscosity

In this section, we derive the Středa formula for the generic electromagnetic formalism. In the case of the usual electromagnetism, this yields the formula expressing the Hall conductivity in terms of bulk quantities [18, 19]. Recently, it has been shown that this formula is also possible for the thermal Hall system [6]. We would like to show that such a useful formula for the Hall viscosity is obtained from viscoelastic-electromagnetism.

We now start with the standard electromagnetism. Let us introduce a magnetization, which is conjugated to the corresponding magnetic field,

$$M_i = -\frac{\delta S}{\delta B^i} = -\frac{\delta S}{\delta A_j} \frac{\partial A_j}{\partial B^i}. \quad (3.1)$$

Recalling the derivative of Lagrangian with respect to the gauge field gives rise to the conserved current, and substituting the expression of the symmetric gauge in the presence of magnetic field,  $A_i = -\epsilon_{ijk} x^j B^k/2$ , the magnetization (3.1) is rewritten as

$$M_i = \frac{1}{2} \epsilon_{ijk} x^j J^k. \quad (3.2)$$

The Hall current is given by using the magnetization:

$$J_H^i = \epsilon^{ijk} \partial_j M_k. \quad (3.3)$$

In order to study the conductivity, we should rewrite this expression in terms of the corresponding electric field. Since the gradient of the chemical potential yields the electric field,  $E_i = \partial_i \mu/e$ , we have the relation between the current and the electric field,

$$J_H^i = \epsilon^{ijk} \left( \frac{\partial M_k}{\partial \mu} \right)_{T,B} \partial_j \mu = \epsilon^{ijk} \left( \frac{\partial M_k}{\partial \mu} \right)_{T,B} e E_j, \quad (3.4)$$

where  $e$  is the electric charge. This means the corresponding Hall transport coefficient is given by

$$\sigma_H = e \left( \frac{\partial M_z}{\partial \mu} \right)_{T,B}. \quad (3.5)$$

The conjugate quantities to the magnetization and the chemical potential are magnetic field  $B$  and particle number density  $n$ , respectively. Thus, by using the thermodynamic relation, as discussed in [19], this Hall coefficient is also written as

$$\sigma_H = e \left( \frac{\partial n}{\partial B} \right)_{T,\mu}. \quad (3.6)$$

This is so-called the Středa formula for the Hall conductivity [18, 19]. While this original formula is of the electric Hall current, its generalization to the thermal transport is also proposed by applying essentially the same procedure to gravito-electromagnetism [6].

Let us then apply this derivation to viscoelastic-electromagnetism based on Sec. 2.2. The corresponding magnetization is similarly introduced as

$$M_{ia} = -\frac{1}{\det e} \frac{\delta S}{\delta B^{ia}} = -\frac{1}{\det e} \frac{\delta S}{\delta e_j^b} \frac{\partial e_j^b}{\partial B^{ia}}. \quad (3.7)$$

We now define the current as the derivative of Lagrangian as well as the other electromagnetisms,

$$J^i_a = \frac{1}{\det e} \frac{\delta S}{\delta e_i^a} = \frac{\partial \mathcal{L}}{\partial e_i^a} + e^i_a \mathcal{L}, \quad (3.8)$$

where  $\mathcal{L}$  is the Lagrangian density. Thus, the corresponding current to viscoelastic theory turns out to be the spatial part of the stress tensor, including both of indices for local and global coordinates,  $J^i_a = T^i_a$ . Substituting the symmetric gauge configuration to this expression,  $e_i^a = \epsilon_{ijk} x^j B^{ka}$ , the magnetization is rewritten as

$$M_{ia} = \frac{1}{2} \epsilon_{ijk} x^j T^k_a. \quad (3.9)$$

This is quite analogous to the gravito-magnetization, which is given by the angular momentum operator,  $M_i = \epsilon_{ijk} x^j T^{k0}/2 = L_i$ . Then, the Hall current in terms of the magnetization is given by

$$J^i_{Ha} = \epsilon^{ijk} \partial_j M_{ka}. \quad (3.10)$$

Then, applying the chain rule, we have the following expression for the stress tensor,

$$J^i_{Ha} = \epsilon^{ijk} \left( \frac{\partial M_{ka}}{\partial v_b} \right)_{T,B} \partial_j v_b = \epsilon^{ijk} \left( \frac{\partial M_{ka}}{\partial v_b} \right)_{T,B} E_{jb}, \quad (3.11)$$

where  $v_b = e^0_b$  is the fluid velocity introduced in Eq. (2.13). Since the coefficient of the gradient of velocity in the stress tensor gives the viscosity tensor, we obtain

$$\eta^{ij}{}_a{}^b = \epsilon^{ijk} \left( \frac{\partial M_{ka}}{\partial v_b} \right)_{T,B}. \quad (3.12)$$

The shear viscosity is symmetric under any exchange of indices. However, this viscosity tensor includes anti-symmetric part, giving rise to dissipationless transport of momentum. The Hall viscosity, which characterizes such a dissipationless transport, is given by the anti-symmetric part of the viscosity tensor (3.12). Therefore, we obtain the Středa formula for the Hall viscosity,

$$\eta_{H,a}{}^b = \left( \frac{\partial M_a}{\partial v_b} \right)_{T,B} = \left( \frac{\partial P^b}{\partial B^a} \right)_{T,v}. \quad (3.13)$$

Note that velocity  $v^b$  is conjugate to momentum  $P^b$ . Usually this part should be diagonal by assuming the isotropic configuration,  $\eta_{H,a}{}^b \propto \delta_a{}^b$ .



This Středa formula can be generalized to (3+1)-dimensional case to obtain the magneto-electric polarizability, which characterizes the cross-response between electric and magnetic fields [6]. In analogy with this we can introduce the analogous quantity, characterizing the relation between viscoelastic-electromagnetic fields, viscoelastic-magnetoelectric polarizability,

$$\chi_{ia}^{jb} = \frac{\partial M_{ia}}{\partial E^{jb}} = \frac{\partial P_{jb}}{\partial B^{ia}}, \quad (3.14)$$

where  $P_{ia}$  stands for the conjugate variable to viscoelastic-electric field. It is quite natural that this quantity describes the cross-correlation between electric and magnetic responses because it is also written as  $\chi = -\partial^2 \mathcal{L} / \partial E \partial B$ .

## 4 Summary and discussion

In this paper, we have introduced the idea of viscoelastic-electromagnetism as a useful method to deal with viscoelastic theory. We have then obtained the Středa formula for the Hall viscosity as an application of such a generalized electromagnetism.

We have shown that the vielbein, which is used in Cartan's formalism of general relativity, can be interpreted as the vector potential for viscoelastic theory. The corresponding electric and magnetic fields have natural meanings in terms of viscoelastic theory, velocity gradient and the Burgers vector, respectively. We have pointed out that the gauge symmetry for this formalism is also reasonable in viscoelastic theory, and the topological action is constructed quite similarly to the usual electromagnetism.

We have then derived the Středa formula for the Hall viscosity. We can follow almost the same procedure to obtain this formula by using viscoelastic-electromagnetic formalism. We have clarified the viscoelastic analog of the magnetization, which is essential for the derivation of the Středa formula. Then, generalizing this result to 3-dimensional theory, we have obtained viscoelastic-magnetoelectric polarizability, which characterizes the cross-correlation between viscoelastic-electric and magnetic responses.

Let us comment on possibilities of future works along this direction. First it is quite interesting to clarify the relation between the Hall conductivity and the Hall viscosity in terms of viscoelastic-electromagnetism [15]. It is shown that the cross-correlation term, called Wen-Zee term [29], plays an essential role in the derivation of such a relation. Thus, we have to somehow involve such an interesting term in electromagnetic formalism. It seems also important to search any other cross-responses between those Hall transport coefficients. Second possibility is extension of the Středa formula to other dissipationless transport. Indeed, the Středa formula for spin Hall conductivity is investigated in [30]. Thus, it is interesting to apply it to spin Hall version of viscosity, which is discussed in [11], as a simple generalization of the formula derived in this paper. Finally, it is possible to apply viscoelastic-electromagnetic formalism to holographic description of the strongly correlated systems, which is based on

AdS/CFT. In particular, it is meaningful to explore its stringy origin by considering D-brane construction.

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